Bertrandův postulát aneb jak hustě jsou rozložena prvočísla?

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Edmund Landau

mezinárodní kongres matematiků 1912

① Goldbachova hypotéza:

"Každé sudé číslo větší než 2 lze vyjádřit jako součet dvou prvočísel."

(1742 v korespondenci mezi Goldbachem a Eulerem)

② Problém prvočíselných dvojčat:

"Je jich ∞ mnoho?"

③ Legendrova hypotéza:

"$\forall n \in \mathbb{N}\ \exists p$ (prvočíslo): $n^2 < p < (n+1)^2$."

④ "$\exists\ \infty$ mnoho prvočísel p tvaru $p = n^2 + 1$ pro nějaké $n \in \mathbb{N}$?"

3 910 693, 3 910 717, 3 910 727, 3 910 733, 3 910 747, 3 910 763, 3 910 787,
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9 248 101, 9 248 111, 9 248 123, 9 248 147, 9 248 159, 9 248 171, 9 248 189,
9 248 201, 9 248 219, 9 248 233, 9 248 243, 9 248 249, 9 248 251, 9 248 273,

Bertrandův postulát:

$$\boxed{\forall n \geq 1 \quad \exists p \text{ (prvočíslo)} : \quad n < p \leq 2n}$$

Joseph Bertrand 1845 (ověřil $\forall n < 3\,000\,000$)
Pafnuty Chebyshev 1850 1. známý důkaz
. . . . . .
Ramanujan
. . . . . .
Paul Erdös 1932 (1. článek v 19ti letech)
elementární důkaz založený na dolním a horním odhadu kombinačního čísla

$$? \leq \binom{2n}{n} \leq ?$$

Myšlenka důkazu B. p. : SPOREM

$$\boxed{\exists\, n \geq 1 : \text{ mezi } n \text{ a } 2n \text{ neleží prvočíslo}}$$

$$\Downarrow$$

$$L = \frac{4^n}{2n} \leq \binom{2n}{n} \leq (2n)^{\sqrt{2n}}\, 4^{\frac{2}{3}n} = P$$

ukážeme, že pro $n > 4000$ platí $L > P$ ↯

Pro $n = 1, 2, \ldots, 4000$ lze B. p. prověřit empiricky
(Eratosthenovo síto, program, …)

p : 2, 3, 5, 7, 13, 23, 43, 83, 163, 317, 631, 1259, 2503, 4001

Out[12]= {2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79,
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ODHAD ZDOLA

$$\forall n \geq 1 : \quad \binom{2n}{n} \geq \frac{4^n}{2n}$$

Důkaz: $4^n = 2^{2n} = (1+1)^{2n} =$

$$= \binom{2n}{0} + \binom{2n}{1} + \dots + \binom{2n}{n} + \dots + \binom{2n}{2n-1} + \binom{2n}{2n}$$

$$= \underbrace{\underbrace{\boxed{\binom{2n}{0} + \binom{2n}{2n}}}_{=2} + \boxed{\binom{2n}{1}} + \dots + \boxed{\binom{2n}{n}} + \dots + \boxed{\binom{2n}{2n-1}}}_{2n \text{ členů}}$$

$$\underbrace{\binom{2n}{n}}_{\text{největší}} \geq \text{průměr} = \frac{2^{2n}}{2n} = \frac{4^n}{2n}.$$

Q.E.D.

ODHAD SHORA

označení: $\binom{2n}{n} = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$

$\sigma_p(a)$ je největší mocnina p, která dělí číslo a

např. $\alpha_1 = \sigma_{p_1}\left(\binom{2n}{n}\right)$ atd.

platí: $\sigma_p(ab) = \sigma_p(a) + \sigma_p(b)$

$$\sigma_p\left(\frac{a}{b}\right) = \sigma_p(a) - \sigma_p(b)$$

$$\binom{2n}{n} = \prod_{p \leq 2n} p^{\sigma_p\left(\binom{2n}{n}\right)}$$

($2n(2n-1)(2n-2)\ldots = \binom{2n}{n} n(n-1)(n-2)\ldots n(n-1)(n-2)\ldots 1$

$\Rightarrow$ prvočíslo p v rozkladu $\binom{2n}{n}$ je $\leq 2n$!)

$$\binom{2n}{n} = \prod_{p \leq 2n} p^{\sigma_p\left(\binom{2n}{n}\right)} =$$

$$= \prod_{p \leq \sqrt{2n}} p^{\sigma_p\left(\binom{2n}{n}\right)} \cdot \prod_{\sqrt{2n} < p \leq \frac{2n}{3}} p^{\sigma_p\left(\binom{2n}{n}\right)}$$

$$\prod_{\frac{2n}{3} < p \leq n} p^{\sigma_p\left(\binom{2n}{n}\right)} \cdot \prod_{n < p \leq 2n} p^{\sigma_p\left(\binom{2n}{n}\right)}$$

předpoklad $\nexists\, n < p \leq 2n \Rightarrow \; = 1$

Legendrova věta: Číslo $N!$ obsahuje prvočíslo p ve svém rozkladu právě

$$\sum_{k \geq 1} \left[\frac{N}{p^k}\right] \quad \text{-krát.}$$

cela část

Důkaz: $N(N-1)(N-2)\ldots 1 = N! = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}$

1. v rozkladu $N!$ je $\left[\frac{N}{p}\right]$ činitelů dělitelných p
2. — " — $\left[\frac{N}{p^2}\right]$ — " — p^2

a ty musíme přidat

atd.

Q.E.D.

$$\boxed{\forall p: \quad p^{\sigma_p\left(\binom{2n}{n}\right)} \leq 2n}$$

Důkaz: $\sigma_p\left(\binom{2n}{n}\right) = \sigma_p(2n!) - \sigma_p(n!n!) =$

$$= \sigma_p(2n!) - 2\sigma_p(n!) = \sum_{k\geq 1}\left[\frac{2n}{p^k}\right] - 2\sum_{k\geq 1}\left[\frac{n}{p^k}\right]$$

Legendrova věta

$$= \sum_{k\geq 1}\Bigg(\underbrace{\left[\frac{2n}{p^k}\right] - 2\left[\frac{n}{p^k}\right]}_{\leq 1}\Bigg) \leq \max\{r : p^r \leq 2n\}$$

$(= 0$ pokud $p^k > 2n)$

$$\left[\frac{2n}{p^k}\right] - 2\left[\frac{n}{p^k}\right] < \frac{2n}{p^k} - 2\left(\frac{n}{p^k} - 1\right) = 2$$

Q.E.D.

JINÝMI SLOVY:

NEJVYŠŠÍ MOCNINA p, KTERÁ DĚLÍ ČÍSLO $\binom{2n}{n}$ JE NEJVÝŠE ROVNA $2n$

DŮSLEDEK: Prvočísla $p > \sqrt{2n}$ se objeví v rozkladu čísla $\binom{2n}{n}$ nejvýše jednou!

DŮLEŽITÉ POZOROVÁNÍ:

Prvočísla $\frac{2n}{3} < p \leq n \quad (n \geq 3)$

se v rozkladu čísla $\binom{2n}{n}$

neobjeví vůbec!

Důkaz: $\frac{2n}{3} < p \Rightarrow 3p > 2n \Rightarrow$

$$\underbrace{2n!}_{\cancel{p}\cdot 2\cancel{p}} = \binom{2n}{n} \cdot \underbrace{n!}_{\cancel{p}}\,\underbrace{n!}_{\cancel{p}}$$

žádné p

Q.E.D.

ČÁSTEČNÉ SHRNUTÍ:

$$\binom{2n}{n} = \underbrace{\prod_{p \le \sqrt{2n}} p^{\overbrace{\sigma_p\left(\binom{2n}{n}\right)}^{\le 2n}}}_{\le (2n)^{\sqrt{2n}}} \cdot \underbrace{\prod_{\sqrt{2n} < p \le \frac{2n}{3}} p^{\sigma_p\left(\binom{2n}{n}\right)}}_{\le ?}$$

$$\underbrace{\prod_{\frac{2n}{3} < p \le n} p^{\sigma_p\left(\binom{2n}{n}\right)}}_{= 1} \cdot \underbrace{\prod_{n < p \le 2n} p^{\sigma_p\left(\binom{2n}{n}\right)}}_{= 1}$$

podle předpokladu

ZBÝVÁ ODHADNOUT

$$\prod_{\sqrt{2n} < p \leq \frac{2n}{3}} p^{\sigma_p\left(\binom{2n}{n}\right)} \quad !$$

$$\boxed{\forall N \geq 1 : \prod_{p \leq N} p \leq 4^N}$$

Důkaz: 1. $N = 1, 2, 3, 4, 5$ platí. Dále mat. indukce vzhledem k N.

2. N je sudé:

$$\prod_{p \leq N} p = \prod_{p \leq N-1} p \underset{\text{(indukční předpoklad)}}{\leq} 4^{N-1} \leq 4^N$$

3. N je liché: $\exists m : N = 2m+1$

$$\prod_{p \le N} p = \prod_{p \le m+1} p \cdot \prod_{m+2 \le p \le 2m+1} p$$

$\le 4^{m+1}$ (indukční předpoklad)

všechna tato prvočísla dělí číslo $\binom{2m+1}{m}$

$\le \binom{2m+1}{m} \le 2^{2m}$

$$\le 4^{m+1} \cdot 2^{2m} = 4^{2m+1} = 4^N$$

$$2^{2m+1} = (1+1)^{2m+1} = \binom{2m+1}{0} + \dots + \binom{2m+1}{m} + \binom{2m+1}{m+1} + \dots + \binom{2m+1}{2m+1}$$

$\binom{2m+1}{m} + \binom{2m+1}{m+1} = 2\binom{2m+1}{m}$

$$\Rightarrow \cancel{2}\binom{2m+1}{m} \le 2^{2m+\cancel{1}}$$

Q.E.D.

SHRNUTÍ:

$$\binom{2n}{n} = \underbrace{\prod_{p \le \sqrt{2n}} p^{\sigma_p\left(\binom{2n}{n}\right)}}_{\le (2n)^{\sqrt{2n}}} \underbrace{\prod_{\sqrt{2n} < p \le \frac{2n}{3}} p^{\sigma_p\left(\binom{2n}{n}\right)}}_{\le 4^{\frac{2n}{3}}}$$

$$\underbrace{\prod_{\frac{2n}{3} < p \le n} p^{\sigma_p\left(\binom{2n}{n}\right)}}_{= 1} \underbrace{\prod_{n < p \le 2n} p^{\sigma_p\left(\binom{2n}{n}\right)}}_{= 1}$$

ODHAD SHORA: $\binom{2n}{n} \le (2n)^{\sqrt{2n}} \cdot 4^{\frac{2n}{3}}$

$$\frac{4^n}{2n} \le \binom{2n}{n} \le (2n)^{\sqrt{2n}} 4^{\frac{2n}{3}}$$

Ukážeme, že

$$\frac{4^n}{2n} \le (2n)^{\sqrt{2n}} 4^{\frac{2n}{3}} \qquad (*)$$

neplatí pro $n \ge 4000$!

Důkaz: $2^k = (1+1)^k = \binom{k}{0} + \binom{k}{1} + \ldots > 1 + k \implies$

$$2n = \left(\sqrt[6]{2n}\right)^6 < \left([\sqrt[6]{2n}] + 1\right)^6 < \left(2^{[\sqrt[6]{2n}]}\right)^6 = 2^{6[\sqrt[6]{2n}]} \le$$

$$\le 2^{6\sqrt[6]{2n}}$$

$$(*) \iff 4^{\frac{n}{3}} \le (2n)^{\sqrt{2n}+1} \iff 2^{2n} \le (2n)^{3(1+\sqrt{2n})}$$

$$2^{2n} \le (2n)^{3(1+\sqrt{2n})} < 2^{6\sqrt[6]{2n}\left[3(1+\sqrt{2n})\right]} =$$

$$= 2^{\sqrt[6]{2n}\,(\underbrace{18}_{<2\sqrt{2n}} + 18\sqrt{2n})} < 2^{20\sqrt[6]{2n}\,\sqrt{2n}} =$$

(pokud $n \ge 50$)

$$= 2^{20(2n)^{2/3}}$$

$$2n < 20\cdot(2n)^{2/3}$$

$$(2n)^{1/3} < 20$$

$$n < 4000 \qquad \lightning$$

Q.E.D.

Děkuji za pozornost!